Connes Embeddings and von Neumann Regular Closures of Group Algebras*

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Abstract

The analytic von Neumann regular closure $R(\Gamma)$ of a complex group algebra $\mathbb{C}\Gamma$ was introduced by Linnell and Schick. This ring is the smallest $*$-regular subring in the algebra of affiliated operators $U(\Gamma)$ containing $\mathbb{C}\Gamma$. We prove that all the algebraic von Neumann regular closures corresponding to sofic representations of an amenable group are isomorphic to $R(\Gamma)$. This result can be viewed as a structural generalization of Lück's Approximation Theorem.
The main tool of the proof which might be of independent interest is that an amenable group algebra $K\Gamma$ over any field K can be embedded to the rank completion of an ultramatricial algebra.

1 Introduction

1.1 Regular rank rings

In this paper all rings are considered unital. Regular rings were introduced by John von Neumann, these are the rings where any principal right ideal is generated by an idempotent (see [8]). A $*$-regular ring $\mathcal{R}$ is a ring with involution and $a^*a = 0$ implies that $a = 0$. In a $*$-regular ring any princial right ideal is generated by a unique projection [10]. A $*$-regular ring $\mathcal{R}$ is *proper* if $\sum_{i=1}^n a_i a_i^* = 0$ implies that all of the a_i's equal to 0. Note that $\mathrm{Mat}_{d\times d}(\mathcal{R})$ is regular if and only if $\mathcal{R}$ is regular, nevertheless for a $*$-regular ring $\mathcal{R}$ $\mathrm{Mat}_{d\times d}(\mathcal{R})$ is $\star$-regular if and only if $\mathcal{R}$ is proper [1]. Since $\mathrm{Mat}_{k\times k}(\mathrm{Mat}_{d\times d}(\mathcal{R})) = \mathrm{Mat}_{kd\times kd}(\mathcal{R})$, $\mathcal{R}$ is proper if and only if all the matrix rings over $\mathcal{R}$ are proper.
A rank function on a regular ring $\mathcal{R}$ is function $\mathrm{rk} : \mathcal{R} \to \mathbb{R}$ satisfying the following conditions.

1. $0 \leq \mathrm{rk}(a) \leq 1$.

2. $\mathrm{rk}(a) = 0$ if and only if $a = 0$.

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3. $\mathrm{rk}(a+b) \leq \mathrm{rk}(a) + \mathrm{rk}(b)$.

4. $\mathrm{rk}(ab) \leq \mathrm{rk}(a), \mathrm{rk}(b)$

5. If e, f are orthogonal idempotents then $\mathrm{rk}(e+f) = \mathrm{rk}(e) + \mathrm{rk}(f)$.

The most important examples of regular rank rings are matrix rings over division rings. In this case, the values of the rank are always rational. The rank defines a metric on the regular ring by $d(x,y) = \mathrm{rk}(x-y)$. The completion of this metric is a regular rank ring as well. Note that for the completion of ultramatricial algebras (see Section 3) the values of the rank can be any real number in between zero and one [8]. Let $\mathcal{N}$ be a finite von Neumann algebra then its Ore localization with respect to its non-zero divisors $U(\mathcal{N})$ is a $\star$-regular ring. The elements of this ring are called affiliated operators (see [16]). The rank of an affiliated operator is the trace of the idempotent that generates the right ideal generated by the operator. Note that if $A \in U(\mathcal{N})$,then

$$\mathrm{rk}(A) = 1 - \lim_{t \to \infty} \int_0^t tr_N(E_\lambda) d\lambda \,, \tag{1}$$

where $\int_0^\infty E_\lambda d\lambda$ is the spectral decomposition of the unbounded operator A^*A. This shows that if $i : \mathcal{N} \to \mathcal{M}$ is a trace-preserving homomorphism between finite von Neumann algebras, then its Ore-extension $\tilde{i} : U(\mathcal{N}) \to U(\mathcal{M})$ is a rank preserving $\star$-homomorphism. Note that $U(\mathcal{N})$ is always proper (see Section 2). If $\mathcal{R}$ is regular rank ring then there is a unique natural extension of the rank to a matrix rank of $\mathrm{Mat}_{k \times k}(\mathcal{R})$ [9]. Note that a matrix rank rkm has the same property as the rank rk except that $0 \leq \mathrm{rkm}(M) \leq k$.

1.2 The Connes Embedding Problem

Let $\nu = \{d_1 < d_2 < \dots\}$ be an infinite sequence of positive integers. Then one can consider the ultraproduct of the matrix algebras $\{\mathrm{Mat}_{d_i \times d_i}(\mathbb{C})\}_{i=1}^\infty$ as tracial algebras the following way (see [15]).

Let ω be a nonprincipal ultrafilter on the natural numbers and let $\lim_\omega$ be the corresponding ultralimit. First, consider the algebra of bounded elements

$$\mathcal{B} = \{(a_1, a_2, \dots) \in \prod_{i=1}^\infty \mathrm{Mat}_{d_i \times d_i}(\mathbb{C}) \mid \sup \|a_i\| < \infty\} \,.$$

Now let $\mathcal{I} \lhd \mathcal{B}$ be the ideal of elements $\{a_i\}_{i=1}^\infty$ such that $\lim_\omega \frac{tr(a_n^* a_n)}{d_n} = 0$. Then $\mathcal{B}/\mathcal{I} = \mathcal{M}_\nu$ is a type II_1-von Neumann factor with trace defined the following way.

$$Tr_\omega[\{a_i\}_{i=1}^\infty] = \lim_\omega \frac{tr(a_n)}{d_n} \,.$$

The following conjecture is generally referred to as the Connes Embedding Problem. Is it true that a type-II_1-von Neumann algebra with a separable

predual have a trace-preserving embedding to some $\mathcal{M}_\nu$? See the survey of Pestov [15] for further details.
There is a purely algebraic version of the Connes Embedding Problem first considered in [6]. Namely, we can consider the ultraproduct of the matrix rings $\{\mathrm{Mat}_{d_i\times d_i}(\mathbb{C})\}$ as *rank algebras*. Let $\mathcal{J} \lhd \prod_{i=1}^\infty \mathrm{Mat}_{d_i\times d_i}(\mathbb{C})$ be the following ideal,

$$\mathcal{J} = \{\, \{a_i\}_{i=1}^\infty \mid lim_\omega \frac{rank(a_n)}{d_n} = 0\}\,.$$

Then $\prod_{i=1}^\infty \mathrm{Mat}_{d_i\times d_i}(\mathbb{C})/\mathcal{J} = \mathcal{M}_\nu^{alg}$ is a simple complete $\star$-regular rank ring. One can ask of course, whether any countable dimensional regular rank ring embeds to some $\mathcal{M}_\nu^{alg}$.

1.3 Lück's Approximation Theorem

Let Γ be a finitely generated residually finite group and let
$\Gamma = N_0 \supset N_1 \supset N_2 \ldots, \cap_{k=1}^\infty N_k = \{1\}$ be finite index normal subgroups. Let $\Delta \in \mathrm{Mat}_{d\times d}(\mathbb{Z}\Gamma)$ be a $d\times d$-matrix over the integer group algebra $\mathbb{Z}\Gamma$. Denote by $\mathcal{N}(\Gamma)$ the von Neumann algebra of Γ. Note that Δ acts on $(l^2(\Gamma)^d)$ as a bounded operator. Then one can define $\dim_\Gamma Ker(\Delta)$, the von Neumann dimension of the kernel of Δ. Let $\pi_k : \mathbb{C}\Gamma \to \mathbb{C}(\Gamma\backslash N_k)$ the natural projection. That is $\pi_k(\Delta) \in \mathrm{Mat}_{d\times d}(\mathbb{C}(\Gamma\backslash N_k))$ is a finite dimensional linear transformation. According to Lück's Approximation Theorem (see [13])

$$\lim_{k\to\infty} \frac{\dim_\mathbb{C} Ker(\pi_k(\Delta))}{|\Gamma : N_k|} = \dim_\Gamma Ker(\Delta) \tag{2}$$

It is conjectured that (2) holds for any $\Delta \in \mathrm{Mat}_{d\times d}(\mathbb{C}\Gamma)$ as well. The conjecture was confirmed for amenable groups Γ in [7].

1.4 Regular Closures

Linnell and Schick [12] proved the following theorem (see Section 2). Let $\mathcal{R}$ be a proper $\star$-regular ring. Then for any subset $T \subseteq \mathcal{R}$ there exists a smallest $\star$-regular subring containing T. We call this ring $R(T,\mathcal{R})$ the regular closure of T in $\mathcal{R}$. Let Γ be a countable group, then one can consider the natural embedding of its complex group algebra to its von Neumann algebra $\mathbb{C}(\Gamma) \to \mathcal{N}(\Gamma)$. Let $\mathcal{U}(\Gamma)$ be Ore localization of $\mathcal{N}(\Gamma)$. Then $\mathcal{U}(\Gamma)$ is a proper $\star$-regular ring (see [2]. Therefore one can consider the *analytic regular closure* $R(\mathbb{C}(\Gamma),\mathcal{U}(\Gamma)) = R(\Gamma)$. Now let $\Gamma = N_0 \rhd N_1 \rhd \ldots, \cap_{i=1}^\infty N_i = \{1\}$ be normal subgroups of a residually finite group.Let $\pi_i : \mathbb{C}\Gamma \to \mathbb{C}(\Gamma/N_i)$ be the natural projection as in the previous subsection and let $s_i : \mathbb{C}(\Gamma/N_i) \to \mathrm{Mat}_{\Gamma/N_i\times\Gamma/N_i}(\mathbb{C})$ be the natural representation by convolutions. Define $r_i = s_i \circ \pi_i : \mathbb{C}\Gamma \to \mathrm{Mat}_{\Gamma/N_i\times\Gamma/N_i}(\mathbb{C})$. Then we have an injective (see [6]) $\star$-homomorphism $r : \mathbb{C}\Gamma \to \mathcal{M}_\nu^{alg}$, where $\nu = \{|\Gamma/N_1|, |\Gamma/N_2|, \ldots\}$. Therefore we can consider the *algebraic regular closure* $R(\mathbb{C}\Gamma, \mathcal{M}_\nu^{alg})$ for any normal chain of residually finite group. The main result of this paper is the following theorem.

Theorem 1. *Let Γ be a finitely generated amenable group. Then there is a rank preserving $\star$-homomorphism*

$$j : R(\Gamma) \to R(\mathbb{C}\Gamma, \mathcal{M}_\nu^{alg})$$

which is the identity map restricted on $\mathbb{C}\Gamma$.

This theorem can be viewed as a structural generalization of Lück's Approximation Theorem for amenable groups. Indeed, let $\Delta \in \mathrm{Mat}_{k\times k}(\mathbb{C}\Gamma)$ then the approximation theorem is equivalent to the fact that

$$\mathrm{rkm}_1(\Delta) = \mathrm{rkm}_2(\Delta)\,,$$

where rkm_1 resp. rkm_2 are the matrix rank on $\mathrm{Mat}_{k\times k}(U(\Gamma))$ resp. on $\mathrm{Mat}_{k\times k}(\mathcal{M}_\nu^{alg})$. However, both $\mathrm{rkm}_1(\Delta)$ and $\mathrm{rkm}_2(\Delta)$ are equal to the matrix rank of Δ in $\mathrm{Mat}_{k\times k}(R(\Gamma))$. Actually, we prove a generalization of Theorem 1, where we consider algebraic closures associated to arbitrary sofic representations (see Section 6) of the group Γ. We shall also prove the following theorem.

Theorem 2. *For any finitely generated amenable group and coefficient field K, the group algebra $K\Gamma$ embeds to the rank completion of an ultramatricial algebra.*

2 von Neumann regular closures

In this section, we review some results of Linnell and Schick [11], [12] about the von Neumann regular closures in proper $\star$-regular rings. The starting points of Linnell's paper are the following two observations about finite von Neumann algebras already mentioned in the introduction.

1. $U(\mathcal{N})$ is a $\star$-regular ring, that is any right ideal is generated by a single projection.
2. If $\alpha, \beta \in U(\mathcal{N})$ and $\alpha\alpha^* + \beta\beta^* = 0$ then $\alpha = \beta = 0$.

Although Linnell and Schick consider only group von Neumann algebras all what they used are the two properties above. The following result is a strengthening of the second observation.

Proposition 2.1 (Lemma 2. [11])**.** *If $\alpha, \beta \in U(\mathcal{N})$ then $(\alpha\alpha^* + \beta\beta^*)U(\mathcal{N}) \supseteq \alpha U(\mathcal{N})$.*

Using a simple induction one also has the following proposition.

Proposition 2.2 (Lemma 2.5 [12])**.** *If $\alpha_1, \alpha_2, \dots, \alpha_n \in U(\mathcal{N})$ then*

$$\sum_{i=1}^n \alpha_i\alpha_i^* U(\mathcal{N}) \supseteq \alpha_1 U(\mathcal{N})\,.$$

This leads to the crucial proposition about the existence of the von Neumann regular closures.

Proposition 2.3 (Proposition 3.1 [12]). *Let $\{R_i \mid i \in I\}$ be a collection of $\star$-regular subrings of $U(\mathcal{N})$. Then $\cap_{i\in I} R_i$ is also a $\star$-regular subring of $U(\mathcal{N})$.*

We also need to show that the proposition above holds for $\mathcal{M}_\mu^{alg}$ as well.

Proposition 2.4. *Let $\{R_i \mid i \in I\}$ be a collection of $\star$-regular subrings of $\mathcal{M}_\mu^{alg}$. Then $\cap_{i\in I} R_i$ is also a $\star$-regular subring of $\mathcal{M}_\mu^{alg}$.*

Proof. Since $\mathcal{M}_\mu^{alg}$ is a $\star$-regular ring we only need to prove that if $\alpha\alpha^* + \beta\beta^* = 0$ in $\mathcal{M}_\mu^{alg}$, then both α and β equal to 0. Then the proof of Proposition 2.3 works without any change.

Lemma 2.1. *For finite dimensional matrices $A, B \in Mat_{k\times k}(\mathbb{C})$*

$$\mathit{rank}(AA^* + BB^*) \geq \max(\mathit{rank}(A), \mathit{rank}(B))$$

Proof. If $(AA^* + BB^*)(v) = 0$ then $A^*(v) = 0$ and $B^*(v) = 0$. Hence

$$\ker(AA^* + BB^*) \subseteq \ker(A^*) \cap \ker(B^*)\,.$$

Therefore $\operatorname{rank}(AA^* + BB^*) \geq \operatorname{rank}(A^*) = \operatorname{rank}(A)$ and $\operatorname{rank}(AA^* + BB^*) \geq \operatorname{rank}(B^*) = \operatorname{rank}(B)$ $\square$

Now let $A_n, B_n \in \operatorname{Mat}_{d_n\times d_n}(\mathbb{C})$, then

$$\lim_\omega \frac{\operatorname{rank}(A_nA_n^* + B_nB_n^*)}{d_n} \geq \lim_\omega \frac{\operatorname{rank}(A_n)}{d_n}$$

and

$$\lim_\omega \frac{\operatorname{rank}(A_nA_n^* + B_nB_n^*)}{d_n} \geq \lim_\omega \frac{\operatorname{rank}(B_n)}{d_n}\,.$$

Hence the proposition follows.

3 Bratteli Diagrams, Ultramatricial Algebras and Tilings

Recall that a Bratteli diagram is an oriented countable graph such that the vertex set is partitioned into finite sets $\{Z_i\}_{i=1}^\infty$ such a way that

- If the starting vertex of an edge is Z_i, then the end vertex is necessarily in Z_{i+1}.
- Each vertex has at least one outgoing edge.
- Each vertex α has a non-negative *size* $S(\alpha)$.
- Each edge (from a vertex α to a vertex β) has a non-negative multiplicity $K(\alpha, \beta)$ such that for each $\beta \in Z_{n+1}$, $S(\beta) = \sum_{\alpha\in Z_n} S(\alpha)K(\alpha, \beta)$

Let P_n be a probability distribution function on Z_n. We call the system $\{P_n\}_{n=1}^{\infty}$ a *harmonic function* if

$$P_n(\alpha) = \sum_{\beta \in Z_{n+1}} \frac{S(\alpha)K(\alpha,\beta)}{S(\beta)} P_n(\beta)$$

for any $n \geq 1$ and $\alpha \in Z_n$. An *ultramatricial algebra* is constructed the following way. For each $n \geq 1$ one consider a product ring $\oplus_{l=1}^{i_n} \mathrm{Mat}_{d_l^n \times d_l^n}(\mathbb{C})$. Let $K(d_l^n, d_j^{n+1})$ be non-negative integers satisfying

$$d_j^{n+1} = \sum_{1 \leq l \leq i_n} d_j^n K(d_l^n, d_j^{n+1})$$

for any $n \geq 1$ and $1 \leq j \leq i_{n+1}$.

Now for any $n \geq 1$ and $1 \leq l \leq i_{n+1}$ choose a diagonal embedding

$$E_{n,l} : \oplus_{l=1}^{i_n} (\mathrm{Mat}_{d_l^n \times d_l^n}(\mathbb{C}))^{K(d_l^n, d_j^{n+1})} \to \mathrm{Mat}_{d_j^{n+1} \times d_j^{n+1}}(\mathbb{C})\,.$$

The embeddings define injective maps

$$\phi_n : \oplus_{l=1}^{i_n} \mathrm{Mat}_{d_l^n \times d_l^n}(\mathbb{C}) \to \oplus_{j=1}^{i_{n+1}} \mathrm{Mat}_{d_j^{n+1} \times d_j^{n+1}}(\mathbb{C})\,.$$

The direct limit $\lim_{\rightarrow} \phi_n$ is the ultramatricial algebra $\mathcal{A}_\phi$. Clearly, $\mathcal{A}_\phi$ is a $\star$-regular ring.
Now for any $n \geq 1$ and $1 \leq l \leq i_n$ let $P(d_l^n)$ be real numbers satisfying

$$\sum_{l=1}^{i_n} P(d_l^n) = 1 \tag{3}$$

and

$$P(d_l^n) = \sum_{j=1}^{i_{n+1}} \frac{d_l^n K(d_l^n, d_j^{n+1})}{d_j^{n+1}} P(d_j^{n+1})\,. \tag{4}$$

Then we have a Bratteli diagram with a harmonic function, where the vertices in Z_n are exactly $\{\mathrm{Mat}_{d_l^n \times d_l^n}(\mathbb{C})\}_{l=1}^{i_n}$, with sizes $\{d_l^n\}_{l=1}^{i_n}$. The Bratteli diagram defines a rank function rk_ϕ on $\mathcal{A}_\phi$. Namely, let

$$\mathrm{rk}_\phi(a_1 \oplus a_2 \oplus \cdots \oplus a_{i_n}) = \sum_{l=1}^{i_n} m(a_l) \frac{\mathrm{rank}(a_l)}{d_l^n},$$

where $m(a_l) = P(d_l^n)$ and $\mathrm{rank}(a_l)$ is the rank of the matrix a_l. Then it is easy to see that each ϕ_n is a rank preserving $\star$-isomorphism. Therefore $\mathcal{A}_\phi$ is a rank regular ring.
Now let Γ be a finitely generated group with a symmetric generating system S. The Cayley graph of Γ, $Cay(\Gamma, S)$ is defined as follows.

- $V(Cay(\Gamma, S)) = \Gamma$

- $(a, b) \in E(Cay(\Gamma, S))$ if $as = b$ for some $s \in S$.

Let $F \subset \Gamma$ be a finite set. Then ∂F is the set of vertices that are adjacent to a vertex in the complement of F. The isoperimetric constant of F is defined as

$$i(F) := \frac{|\partial F|}{|F|}\,.$$

The group Γ is amenable if there exists a Følner-sequence in Γ that is a a sequence of finite sets $\{F_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} i(F_n) = 0$.
Now we define Bratteli-tiling systems. If $\gamma \in \Gamma, F \subset \Gamma$ then γF is called a F-tile. A Bratteli system has the following properties.

- The level set Z_n consists of finite sets $F_1^n, F_2^n, \ldots, F_{i_n}^n$ and the set E_n containing only the unit element. Also, we have $i(F_j^n) \leq \frac{1}{2^n}$ for all j and n.
- For any $n \geq 2$ and $F_j^n \in Z_n$ we have a partition $F_j^n = \cup_{i=1}^{a_{n,j}} \gamma_i A_i$, where $A_i \in Z_{n-1}$. That is we have tiling of F_j^n with the tiles of Z_{n-1}.
- $K(F_l^{n-1}, F_j^n)$ is the number of F_l^{n-1}-tiles in the partition of F_j^n. Also $K(E_{n-1}, F_j^n)$ is the number of E_{n-1}-tiles (single vertices).
- $S(F_j^n) = |F_j^n|$, $S(E_n) = 1$.
- We also suppose that $K(E_{n-1}, F_j^n) \leq \frac{1}{2^{n-1}} |F_j^n|$.

Let $m : \cup_{n=1} \infty Z_n \to \mathbb{R}$ be a harmonic function such that $m(E_n) \to 0$ as $n \to \infty$. Then we call a system above a Bratteli tiling system. Our main technical tool is the following proposition.

Proposition 3.1. *For any amenable group Γ and generating system S we can construct a Bratteli tiling system with the following property. For any $\epsilon > 0$ and $n > 0$ there exist $\delta = \delta_{\varepsilon,n} > 0$ such that if $F \in \Gamma$ is a finite set and $i(F) < \delta$ then one can tile F with translates of the elements Z_n satisfying the following property. If $L \in Z_n$ and T_L^F is the set of points in F covered by a translate of L then*

$$\left| \frac{|T_L^F|}{|F|} - m(L) \right| < \epsilon\,.$$

4 Proof of Proposition 3.1

First, let us recall the notion of ε-quasitilings. Let $Cay(\Gamma, S)$ be the Cayley-graph of an amenable group Γ as above. Let $F \subset \Gamma$ be a finite set and $A_1, A_2, \ldots, A_n$ be subsets of F. We say that $\{A_i\}_{i=1}^n$ ε-cover F if

$$\frac{|\cup_{i=1}^n A_i|}{|F|} > 1 - \varepsilon\,.$$

Also, we call $\{A_i\}_{i=1}^n$ ε-disjoint if there exist disjoint sets $\{B_i\}_{i=1}^n$, $B_i \subset A_i$, such that

$$\frac{|B_i|}{|A_i|} > 1 - \varepsilon\,.$$

The system $\{A_i\}_{i=1}^n$ ε-quasi-tiles F if it both ε-covers F and ε-disjoint. The following result of Ornstein and Weiss [14] is crucial for our proof.

Proposition 4.1 (Quasitiling theorem)**.** *Let $F_1 \subset F_2 \subset \dots$ be a Følner-sequence. Then for any $\varepsilon > 0$ ther exists $\delta > 0$ and a subfamily $F_{n_1} \subset F_{n_2} \subset \dots F_{n_k}$ such that if $i(F) < \delta$ then F can be ε-quasitiled by translates of the F_{n_i}'s.*

Observe that if $i(A) < \varepsilon$ and $B \subset A$, $\frac{|B|}{|A|} > 1 - \varepsilon$, then

$$i(B) < (d+1)\frac{\varepsilon}{1-\varepsilon}\,,$$

where d is the degree of the vertices of $Cay(\Gamma, S)$. Indeed, ∂B is covered by the union of ∂A and the neighbours of the vertices in $A \backslash B$. Thus $|\partial B| \leq |\partial A| + d|A\backslash B|$. Therefore,

$$\frac{|\partial B|}{|B|} \leq \frac{|\partial B|}{|A|(1-\varepsilon)} \leq (d+1)\frac{\varepsilon}{1-\varepsilon}\,.$$

Now using the quasitiling theorem we construct a Bratteli system inductively. Suppose that $\{F_1^m, F_2^m, \dots, F_{i_m}^m\}$ and the decreasing sequence of positive constants $\{\delta_n\}_{i=0}^m$ are already given such a way that

- for any $i \geq 1$ $|\partial F_i^m| < \min(\delta_{n_1}, \frac{1}{2^n})$,
- if $i(F) < \delta_m$ then F can be tiled by translates of the F_i^m's and less than $(1/2^m)|F|$ single points.

Now let $G_1 \subset G_2 \subset \dots$ be a Følner-sequence and $c > 0$ such that if $B \subset G_j$ for some j and $\frac{|B|}{|G_j|} > 1 - c$ then

$$i(B) < \min(\delta_n, 1/2^{n+1})\,.$$

By Proposition 4.1 there exists a family of finite subsets $F_1^{n+1}, F_2^{n+1}, \dots, F_{i_{n+1}}^{n+1}$ (namely subsets of a certain system $G_{n_1}, G_{n_2}, \dots, G_{n_k}$) and a constant δ_{n+1} such that

- for any $i \geq 1$ $|\partial F_i^{n+1}| < \min(\delta_n, \frac{1}{2^{n+1}})$,
- if $i(F) < \delta_{n+1}$ then F can be tiled by translates of the F_i^{n+1}'s and less than $(1/2^{n+1})|F|$ single points.

By the induction above one can obtain a Bratteli system. Now we construct a harmonic function m. Fix a Følner-sequence $H_1 \subset H_2 \subset \dots$. Let $\{\delta_n\}_{n=1}^\infty$ be the constans as above. If

$$\min(\frac{1}{j_{n+1}}, \delta_{j_{n+1}}) \leq i(H_n) < \min(\frac{1}{j_n}, \delta_{j_n})$$

then pick a tiling of H_n by translates of the elements of Z_{j_n} such a way that the number of single vertices is less than $(\frac{1}{2^{j_n}})|H_n|$. Then pick a tiling of the Z_{j_n}-tiles by translates of of Z_{j_n-1} such a way that the number of single vertices in any tile T is less than $(\frac{1}{2^{j_n-1}})|T|$. Inductively, we obtain a tiling of H_n by Z_i-translates for any $1 \leq i \leq j_n$. Note that the number of single vertices used in the Z_i-tiling of H_n is less than

$$(\sum_{k=i}^{j_n} \frac{1}{2^k})|H_n| \leq \frac{1}{2^{i-1}}|H_n|\,.$$

If $A \in Z_i$ then denote by $c_k(A)$ the number of vertices in H_k covered by A-translates and let

$$m_k(A) = \frac{c_k(A)}{|H_k|}\,.$$

Clearly, $\sum_{A\in Z_i} m_k(A) = 1$. We may suppose that for any A, $\lim_{k\to\infty} m_k(A) = m(A)$ exists, otherwise we could pick a subsequence of $\{H_k\}_{k=1}^{\infty}$.

Lemma 4.1. *The function m is harmonic satisfying* $\lim_{i\to\infty} m(E_i) = 0$.

Proof. The fact that $\lim_{i\to\infty} m(E_i) = 0$. follows from our previous observation about the number of single vertices used in the tiling of H_k. By definition, if $A \in Z_i$

$$m_k(A) = \sum_{B\in Z_{i+1}} \frac{S(A)K(A,B)}{S(B)} m_k(B)\,.$$

By taking the limit as $n \to \infty$ we get that

$$m(A) = \sum_{B\in Z_{i+1}} \frac{S(A)K(A,B)}{S(B)} m(B)\,. \quad \square$$

Now let us show that our Bratteli tiling system satisfies the required property. First we prove a simple lemma.

Lemma 4.2. *For any $i > 0$ and $\delta > 0$ there exists $\lambda > 0$ and $p > 0$ with the following property. Let $k > p$ and $J \subseteq H_k$, $\frac{|J|}{|H_k|} > 1 - \lambda$. For $A \in Z_i, |A| > 1$, let j^k_A be the number of vertices in H_k that are covered by an A-translate (in the tiling previously defined) which is completely inside J. Also, let $j^k_{E_i}$ be the number of points in H_k that are not covered by any of the A-translates above. Then*

$$\left|\frac{j^k_A}{H_k} - m(A)\right| < \delta \quad \textit{and} \quad \left|\frac{j^k_{E_i}}{H_k} - m(E_i)\right| < \delta\,.$$

Proof. The number of points covered by such A-translates that contain at least one point from the complement of J is less than $|H_k\backslash J||A|$. Hence

$$m_k(A) \geq j^k_A \geq m_k(A) - \frac{|H_k\backslash J|}{|H_k|}|A|\,.$$

Since $\sup_{A\in Z_i}|A|<\infty$ and $m_k(A)\to m(A)$ the lemma easily follows. $\square$

Now let ε be the constant in our proposition and $0<\alpha<\varepsilon/2$. By Proposition 4.1, we have a subfamily $H_{a_1}, H_{a_2},\dots,H_{a_t}$ of $\{H_k\}_{k=1}^\infty$ that α-quasitiles any finite set F with $i(F)<\delta_\alpha$. By the previous lemma it means that we have disjoint subsets J in F that can be tiled by Z_i-translates such a way that using the notation of our proposition

$$\left|\frac{T_A^F}{|F|}-m(A)\right|<\frac{\varepsilon}{10}\,,$$

for any $A\in Z_i$ provided that α is small enough. We cover all the remaining points (that are not in the J's) by single vertices. Then we get the required tiling of F. $\square$

5 The canonical rank on amenable group algebras

In this section we recall some results from [3]. Let Γ be a finitely generated amenable group and $K\Gamma$ be its group algebra over the field K. Let $\{F_n\}_{n=1}^\infty$ be a Følner- sequence. For $a\in K\Gamma$ let $V_n^a\subset K^{F_n}\subset K\Gamma$ be the vector space of elements z supported on F_n such that $za=0$. Then

$$\lim_{n\to\infty}\frac{\dim_K V_n^a}{|F_n|}=k_a$$

exists and independent on the choice of the Følner-sequence. We call $\mathrm{rk}(a)=1-k_a$ the natural rank of a. It is proved in [5] that if $K=\mathbb{C}$ then $\mathrm{rk}(a)=1-\dim_\Gamma Ker\,M_a$, where dim_Γ is the von Neumann dimension and $Ker\,M_a$ is the set of elements $w\in l^2(\Gamma)$ for which $wa=0$. Note that the rank can be computed slightly differently as well. Let S be a symmetric generating system of Γ and $Cay(\Gamma,S)$ be the Cayley-graph of Γ. We consider the shortest path metric $d_{Cay(\Gamma,S)}$ on Γ. Let $\mathrm{supp}(a)\subset B_r(1)$, where $B_r(1)$ is the r-ball around the unit element in the Cayley-graph and

$$\mathrm{supp}(a)=\{\gamma\in\Gamma\mid a_\gamma\neq 0\}$$

if $a=\sum a_\gamma\gamma$. For a finite set $F\subset\Gamma$, let $\partial_r F$ be the set of elements $x\in F$ such that

$$d_{Cay(\Gamma,S)}(x,F^c)\leq r\,.$$

Note that $\partial F=\partial_1(F)$. Clearly, if $b\in K\Gamma$ and $\mathrm{supp}(b)\in F\backslash\partial_r F$ then $\mathrm{supp}(ba)\subset F$. Then for any $s>r$

$$\mathrm{rk}(a)=\lim_{n\to\infty}\frac{\dim_K W_n a}{|F_n|},\tag{5}$$

where W_n is the set of elements in $K\Gamma$ supported on $F_n\backslash\partial_s F_n$.

6 Sofic representations

6.1 Sofic approximation and sofic representations

In this section we recall the notion of sofic representations from [6]. Let Γ be a finitely generated group with a symmetric generating set S. Let $\{G_n\}_{n=1}^\infty$ be a sequence of finite graphs such that the directed edges are labeled by the elements of S such a way that if (x,y) is labeled by s, then (y,x) is labeled by s^{-1}. We say that $\{G_n\}_{n=1}^\infty$ is a sofic approximation of Γ if for any natural number $r>0$ there exists $n_r>0$ such that

- if $n \geq n_r$ then for the set V_n^r of vertices x for which the ball $B_r(x)$ in G_n is isomorphic to the ball $B_r(1) \in Cay(\Gamma, S)$ as labeled graphs
$$\frac{|V_n^r|}{|V(G_n)|} > 1 - \frac{1}{r}\,.$$

A group is called *sofic* if it possesses a sofic approximation. In this moment no non-sofic group is known. If $Cay(\Gamma,S)$ is the Cayley-graph on an amenable group and $\{F_n\}_{n=1}^\infty$ is a Følner-sequence then the induced graphs G_n of the sets F_n form a sofic approximation of Γ. If Γ is residually finite (amenable or not) with normal chain $\{N_k\}_{k=1}^\infty$, $\cap_{k=1}^\infty N_k = \{1\}$ then the graph sequence $Cay(\Gamma/N_k, S)$ form a sofic approximation of Γ. If $\{G_n\}_{n=1}^\infty$ is an arbitrary sofic approximation of a group Γ then one can construct an imbedding of $K\Gamma$ (K is an arbitrary field) to the ultraproduct of matrix algebras the following way.
Let $\{\mathrm{Mat}_{V(G_n)\times V(G_n)}(K)\}_{n=1}^\infty$ be a sequence of matrix algebras. Let $a \in K\Gamma$, $a = \sum r_\gamma \gamma$ be an element of the group algebra such that if $r_\gamma \neq 0$ then $\gamma \in B_r(1) \subset Cay(\Gamma, S)$. Let $\{e_x\}_{x\in V(G_n)}$ be the natural basis of $K^{V(G_n)}$. If $x \in V_n^r$ then let
$$\psi_n(a)(e_x) = \sum_{y \in B_r(x)} k_y e_y\,,$$
where $k_y = r_\gamma$ if $x\gamma = y$. Note that by our condition on the support of a $x\gamma = y$ is meaningful. If $x \notin V_n^r$ let $\psi_n(a)(x) = 0$. This way one can define an injective homomorphism $\psi_\mu : K\Gamma \to \mathcal{M}_\mu^{alg}$, where $\mu = \{|V(G_1)|, |V(G_2)|, \dots\}$ If $K = \mathbb{C}$ the homomorphism above is a $*$-homomorphism. The map ψ_μ is called the sofic representation associated to the sequence $\{G_n\}_{n=1}^\infty$.

6.2 Sofic approximation of amenable groups

Let $\{G_n\}_{n=1}^\infty$ be a sofic approximation of the amenable group Γ (with symmetric generating set S). For $L>0$ let $Q_L^{G_n}$ be the set of vertices x in G_n such that
$$B_L(x) \cong B_L(1) \subset Cay(\Gamma, S)$$
as S-labeled graphs. If $F \subset B_L(1)$ then for $x \in Q_L^{G_n}$ we call $\pi(F)$ an F-translate, where $\pi : B_L(1) \to B_L(x)$ is the S-labeled graph isomorphism mapping 1 to x. In [7] we proved the following generalization of the Ornstein-Weiss quasitiling theorem.

Proposition 6.1. *Let $F_1 \subset F_2 \dots$ be a Følner-sequence. Then for any $\varepsilon > 0$ there exists $L > 0$, $\delta > 0$ and a finite subcollection $F = \{F_{n_1}, F_{n_2}, \dots, F_{n_t}\}$ such that if*

$$\frac{Q_L^{G_n}}{|V(G_n)|} > 1 - \delta$$

then G_n can be ε-quasitiled by F-translates.

7 Imbedding $K\Gamma$ to the completion of an ultramatricial algebra

Let $(\{Z_i\}_{i=1}^\infty, m)$ be the Bratteli tiling system as in Proposition 3.1. We construct an ultramatricial algebra as in Section 3. Let $\oplus_{A \in Z_i} \mathrm{Mat}_{|A| \times |A|}(K)$ be the i-th algebra. For $B \in Z_{i+1}$ let

$$E_B : \oplus_{A \in Z_i} \mathrm{Mat}_{|A| \times |A|}(K) \to \mathrm{Mat}_{|B| \times |B|}(K)$$

be the diagonal embedding, where the image of each $\mathrm{Mat}_{|A| \times |A|}(K)$ is $K(A, B)$ $|A| \times |A|$-diagonal block in $\mathrm{Mat}_{|B| \times |B|}(K)$. Let

$$\phi_i = \oplus_{B \in Z_{i+1}} E_B : \mathrm{Mat}_{|A| \times |A|}(K) \to \oplus_{B \in Z_{i+1}} \mathrm{Mat}_{|B| \times |B|}(K)$$

the product map. Now we define the maps $\pi_i^A : K\Gamma \to \mathrm{Mat}_{|A| \times |A|}(K)$ the following way. We identify the elements of $\mathrm{Mat}_{|A| \times |A|}(K)$ with the linear transformations from K^A to K^A the natural way. Let $a \in K\Gamma$, $\mathrm{supp}(a) \subset B_r(1)$. If $x \in A \backslash \partial_r(A)$, then let

$$\pi_i^A(a)(e_x) = \sum a_\gamma e_{x\gamma} \,.$$

Note that by the condition on the support $x\gamma$ is well-defined. If $\partial_r(A)$, then let $\pi_i^A(a)(e_x) = 0$. Finally we define the maps $\pi_i := \oplus_{A \in Z_i} \pi_i^A : K\Gamma \to \oplus_{A \in Z_i} \mathrm{Mat}_{|A| \times |A|}(K)$.

Lemma 7.1. *For any $a \in K\Gamma$, $\{[\pi_i(a)]\}_{i=1}^\infty$ is a Cauchy-sequence in $\mathcal{A}_\phi$, where $[\pi_i(a)]$ denotes the image of $\pi_i(a)$ under the natural embedding $\oplus_{A \in Z_i} Mat_{|A| \times |A|}(K) \to \mathcal{A}_\phi$.*

Proof. First of all note that

$$\mathrm{rk}_\phi(\phi_i \circ \pi_i(a) - \pi_{i+1}(a)) = \sum_{B \in Z_{i+1}} m(B) \frac{\mathrm{rank}(E_B \circ \pi_i(a) - \pi_{i+1}^B(a))}{|B|} \,.$$

Observe that

$$\mathrm{rank}(E_B \circ \pi_i(a) - \pi_{i+1}^B(a)) = |B| - \dim_K \ker(E_B \circ \pi_i(a) - \pi_{i+1}^B(a)).$$

On the other hand,

$$\dim_K \ker(E_B \circ \pi_i(a) - \pi_{i+1}^B(a)) \leq T_B \,,$$

where T_B is the number of vertices in B for which

$$E_B \circ \pi_i(a)(e_x) = \pi_{i+1}^B(a)(e_x)\,.$$

Clearly,

$$T_B \geq |B| - |\partial_r B| - \sum_{A\in Z_i} K(A,B)|\partial_r(A)|\,.$$

Recall that if $|B|>1$ then $|\partial B| \leq |B|2^{-(i+1)}$. Hence

$$|\partial_r B| \leq |B|2^{-(i+1)}(d+1)^{r+1}\,,$$

where d is the degree of the vertices in the Cayley graph. Also,

$$\sum_{A\in Z_i} K(A,B)|\partial_r(A)| \leq K(E_i,B) + \sum_{A\in Z_i\,,|A|>1} K(A,B)|A|2^{-i}(d+1)^{r+1}\,.$$

Therefore,

$$T_B \geq |B| - |B|2^{-(i+1)}(d+1)^{r+1} - |B|2^{-(i+1)} - |B|2^{-i}(d+1)^{r+1}\,.$$

Hence,

$$\begin{aligned}&\operatorname{rk}_\phi(\phi_i\circ\pi_i(a) - \pi_{i+1}(a)) \leq \\ &\leq 2^{-(i+1)} + \sum_{B\in Z_{i+1}\,,|B|>1} m(B)(2^{-(i+1)}(d+1)^{r+1} + 2^{-(i+1)} + 2^{-i}(d+1)^{r+1})\end{aligned}$$

Thus the lemma follows. □

Lemma 7.2. *Let $a\in K\Gamma, b\in K\Gamma$, then*

1. *$lim_{i\to\infty} rk_\phi(\phi_i(a)\phi_i(b) - \phi_i(ab)) = 0$.*
2. *$lim_{i\to\infty} rk_\phi(\phi_i(a) + \phi_i(b) - \phi_i(a+b)) = 0$.*
3. *If $K=\mathbb{C}$ then $lim_{i\to\infty} rk_\phi(\phi_i(a^*) - \phi_i^*(a)) = 0$.*

Proof. We prove the first part only, the other two statements can be seen exactly the same way. If $x\in A\backslash\partial_{r+s}(A)$ then

$$\phi_i(a)\phi(b) - \phi_i(ab)(e_x) = 0\,.$$

Therefore

$$\operatorname{rk}_\phi(\phi_i(a)\phi_i(b) - \phi_i(ab)) \leq \frac{|\partial_{r+s}(A)|}{|A|}\,.$$

Hence the lemma follows. □

Let $\phi(a)\in\overline{\mathcal{A}}_\phi$ be the limit of elements $\phi_i(a)$. By the previous lemma ϕ is a homomorphism and if $K=\mathbb{C}$ then ϕ is even a $\star$-homomorphism. Finally, we prove that $\operatorname{rk}_\phi(\phi(a)) = \operatorname{rk}_\Gamma(a)$. By definition,

$$\operatorname{rk}_\phi(\phi_i(a)) = \sum_{A\in Z_i} m(A)\frac{\operatorname{rank}(\phi_i(a))}{|A|}\,.$$

If $|A| = 1$, then $m(A) \leq 2^{-i}$ otherwise by (5)

$$\lim_{i\to\infty} \frac{\mathrm{rank}(\phi_i(a))}{|A|} = \mathrm{rk}_\Gamma(a)\,.$$

Hence, $\mathrm{rk}_\phi(\phi(a)) = \mathrm{rk}_\Gamma(a)$. This finishes the proof of our theorem. $\square$

8 The proof of the main theorem

8.1 The strategy of the proof

We have four complete regular $*$-rings: $\overline{\mathcal{A}_\phi}$, $U(\Gamma)$, $\mathcal{M}_\mu^{alg}$ and $U(\mathcal{M}_\mu)$. Also, we define seven rank preserving embeddings

1. $f_1 : \mathbb{C}\Gamma \to \mathcal{M}_\mu^{alg}$
2. $f_2 : \mathbb{C}\Gamma \to U(\Gamma)$
3. $f_3 : \mathbb{C}\Gamma \to U(\mathcal{M}_\mu)$
4. $f_4 : \mathbb{C}\Gamma \to \overline{\mathcal{A}_\phi}$
5. $f_5 : \overline{\mathcal{A}_\phi} \to \mathcal{M}_\mu^{alg}$
6. $f_6 : \overline{\mathcal{A}_\phi} \to U(\mathcal{M}_\mu)$
7. $f_7 : U(\Gamma) \to U(\mathcal{M}_\mu)$.

We shall show three identities:

1. $f_5 \circ f_4 = f_1$
2. $f_6 \circ f_4 = f_3$.
3. $f_7 \circ f_2 = f_3$

From these identities the main theorem easily follows. Indeed, $R(\Gamma)$ is the smallest $\star$-regular ring containing $\mathbb{C}\Gamma$ in $U(\Gamma)$. The ring $R(\Gamma)$ is inside $\overline{\mathcal{A}_\phi}$, in fact, it is the minimal $\star$-regular ring containing $\mathbb{C}\Gamma$ in $\overline{\mathcal{A}_\phi}$. On the other hand, $R(\mathbb{C}\Gamma, \mathcal{M}_\mu^{alg})$ is also the smallest $\star$-regular ring containing $\mathbb{C}\Gamma$ in $\overline{\mathcal{A}_\phi}$. $\square$

8.2 The first identity

Let $\{G_n\}_{n=1}^\infty$ be the sofic approximation of our group Γ and $\mathcal{M}_\mu^{alg}$ be the associated ultraproduct. Let $\{H_n\}_{n=1}^\infty$ be the Følner-sequence in the proof of Proposition 3.1. Let f_4 be the map $\phi : \mathbb{C}\Gamma \to \overline{\mathcal{A}_\phi}$ defined in the proof of Theorem 2. Let f_1 be the map $\psi_\mu : \mathbb{C}\Gamma \to \mathcal{M}_\mu^{alg}$ defined in Subsection 6.1. Fix $k \geq 1$. Now we define maps $\tau_{k,n} : \oplus_{A\in Z_k}\mathrm{Mat}_{A\times A}(\mathbb{C}) \to \mathrm{Mat}_{V(G_n)\times V(G_n)}(\mathbb{C})$ for large enough $n \geq 1$.

First, let $q \geq 1$ be an integer. We say that G_n is q-regular if G can be $\frac{1}{2^q}$-quasitiled by translates of $\{H_{n_1}, H_{n_2}, \dots H_{n_t}\}$, where $n_i \geq q$. For $n \geq 1$, let $q(n)$ be the largest q for which G_n is q-regular. By Proposition 6.1, for any $q \geq 1$ there exists some n_q such that if $n \geq n_q$ then $q(n) \geq q$.
Now consider a $\frac{1}{2^q}$-quasitiling of G_n by translates of $\{H_{n_1}, H_{n_2}, \dots H_{n_t}\}$. Then consider the iterated tiling for each H_{n_i} above by Z'_ks as in the proof of Proposition 3.1, starting with $Z_{l(n)}$-tilings. Since the translates are not disjoint this does not yet define a tiling of G_n. However, let $\{J_\alpha\}_{\alpha \in I}$ be the disjoint parts in the quasitiling. That is each J_α is inside some H_{n_i}-translate having size at least $(1 - \frac{1}{2^q})|H_{n_i}|$. Discard the tiles that are inside some $Z_{l(n)}$-tile that is not contained in some J_α. Cover, the remaining part of G_n by single vertices. For $A \in Z_k$, let $Q_n(A)$ be the number of vertices in $V(G_n)$ that are covered by an A-translate. By Lemma 4.2, it is easy to see that

$$\lim_{n \to \infty} \frac{Q_n(A)}{|V(G_n)|} = m(A)\,.$$

Now let $\tau_{k,n} : \oplus_{A \in Z_k} \mathrm{Mat}_{A \times A}(\mathbb{C}) \to \mathrm{Mat}_{V(G_n) \times V(G_n)}(\mathbb{C})$ be the natural diagonal map induced by the tiling above. If $|A| > 1$, the definition is clear. The case where $A = E_k$, that is A is a single point needs some clarification. In the diagonal map, we use only those vertices in G_n that are in some "good" $Z_{l(n)}$-tile, in other words, that are not used to cover the remaining part. All the diagonal elements in the image of $\tau_{k,n}$ that belong to a vertex covering the remaining part are zero.
By the iterative tiling construction, one can immediately see that $\tau_{k,n} \circ \phi_k = \tau_{k+1,n}$. If $k > q(n)$, let us define $\tau_{k,n} := 0$. Therefore we have a map

$$\tau = (\tau_1, \tau_2 \dots) : \oplus_{A \in Z_k} \mathrm{Mat}_{A \times A}(\mathbb{C}) \to \oplus_{n=1}^{\infty} \mathrm{Mat}_{V(G_n) \times V(G_n)}(\mathbb{C})$$

and this map extends to $\mathcal{A}_\phi$ as well.

Lemma 8.1. *For any $(a_1 \oplus a_2 \oplus \dots a_{i_k}) \in \oplus_{A \in Z_i} Mat_{A \times A}(\mathbb{C})$*

$$\lim_{n \to \infty} \frac{rank(\tau_{k,n}(a_1 \oplus a_2 \oplus \dots a_{i_k}))}{|V(G_n)|} = rk_\phi(a_1 \oplus a_2 \oplus \dots a_{i_k})\,.$$

Proof.

$$\lim_{n \to \infty} \frac{\mathrm{rank}(\tau_{k,n}(a_1 \oplus a_2 \oplus \dots a_{i_k}))}{|V(G_n)|} = \lim_{n \to \infty} \sum_{A \in Z_k} \frac{\frac{Q_n(A)}{|A|}\mathrm{rank}(a_A)}{|V(G_n)|} =$$

$$= \sum_{A \in Z_k} m(A) \frac{rank(a_A)}{|A|} = \mathrm{rk}_\phi(a_1 \oplus a_2 \oplus \dots a_{i_k})\,.$$

□

Therefore we have a rank-preserving map $\tau_{alg} : \oplus_{A \in Z_k} \mathrm{Mat}_{A \times A}(\mathbb{C}) \to \mathcal{M}_\mu^{alg}$ defined as $\pi \circ \tau_{alg}$, where

$$\pi : \oplus_{n=1}^{\infty} \mathrm{Mat}_{V(G_n) \times V(G_n)}(\mathbb{C}) \to \mathcal{M}_\mu^{alg}$$

is the quotient map. The map τ_{alg} extends to the rank completion of $\mathcal{A}_\phi$, resulting in the map f_5.
Now let us prove the first identity. Let $a \in \mathbb{C}\Gamma$, $\mathrm{supp}(a) \subset B_r(1) \subset Cay(\Gamma, S)$. Then $f_1(a)$ can be represented in $\oplus_{n=1}^{\infty}\mathrm{Mat}_{V(G_n)\times V(G_n)}(\mathbb{C})$ by the element $\oplus_{n=1}^{\infty}\psi_n(a)$, where ψ_n is defined in Subsection 6.1. On the other hand, $f_5 \circ f_4(a)$ is represented by $\oplus_{n=1}^{\infty}\psi'_n(a)$, where

$$\psi'_n(a)(e_x) = \sum_{y\in B_r(x)} k_y e_y \,,$$

where $k_y = r_\gamma$ if $x\gamma = y$ and $x \in \partial_r(J_\alpha)$, for some J_α in a H_{n_i}-translate. Clearly,

$$\lim_{n\to\infty} \frac{z_n(a)}{|V(G_n)|}\,,$$

where $z_n(a)$ is the number of elements $x \in V(G_n)$ for which

$$\psi'_n(a)(e_x) = \psi_n(a)(e_x)\,.$$

Therefore $f_5 \circ f_4 = f_1$.

8.3 The second identity

Let rk_1 resp. rk_2 denote the ranks on $\mathcal{M}_\mu$ resp. $\mathcal{M}_\mu^{alg}$ Let

$$t \in \oplus_{n=1}^{\infty}\mathrm{Mat}_{V(G_n)\times V(G_n)}(\mathbb{C}) = (t_1, t_2, \dots)\,,$$

where $\sup \|t_i\| < \infty$. Note that t represents and element $[t]_{\mathcal{M}_\mu}$ in $\mathcal{M}_\mu$ and an element $[t]_{\mathcal{M}_\mu^{alg}}$ in $\mathcal{M}_\mu^{alg}$. It is important to note that $rk_1([t]_{\mathcal{M}_\mu})$ is not necessarily equal to $rk_2([t]_{\mathcal{M}_\mu^{alg}})$. Indeed, let $t_n = \frac{1}{n}Id$. Then $\mathrm{rk}_1([t]_{\mathcal{M}_\mu}) = 0$. Nevertheless, $rk_2([t]_{\mathcal{M}_\mu^{alg}}) = 1$. However, we have the following lemma.

Lemma 8.2. *Let $t = (t_1, t_2, \dots) \in \oplus_{n=1}^{\infty} Mat_{V(G_n)\times V(G_n)}(\mathbb{C})$, where for any $n \geq 1$, t_n is self-adjoint and all the t_n's have altogether finitely many distinct eigenvalues $\lambda_0 = 0, \lambda_1, \lambda_2, \dots, \lambda_k$. Then*

$$rk_1([t]_{\mathcal{M}_\mu}) = rk_2([t]_{\mathcal{M}_\mu^{alg}})\,.$$

Proof. Let $t_{n,i}$ be the multiplicity of λ_i in t_n. Then

$$\mathrm{rk}_2([t]_{\mathcal{M}_\mu^{alg}}) = \lim_\omega (1 - \frac{t_{n,0}}{|V(G_n)|})\,.$$

The spectral decomposition of $[t]_{\mathcal{M}_\mu}$ is $\sum_{i=1}^k \lambda_i P_i$, where

$$tr_{\mathcal{M}_\mu}(P_i) = \lim_\omega (1 - \frac{t_{n,i}}{|V(G_n)|})\,.$$

By (1)

$$\mathrm{rk}_1([t]_{\mathcal{M}_\mu}) = \lim_\omega (1 - \frac{t_{n,0}}{|V(G_n)|})\,. \qquad \square$$

We also need the following lemma.

Lemma 8.3. *Let t be as above and suppose that*

$$\lim_{n\to\infty} \frac{rank(t_n)}{|V(G_n)|} = 0\,.$$

Then $[t]_{\mathcal{M}_\mu} = 0\,.$

Proof. We need to check that $\lim_{n\to\infty} \frac{tr(t_n^* t_n)}{|V(G_n)|} = 0\,.$ Observe that $\sup \|t_n^* t_n\| = K < \infty$ and $\operatorname{rank}(t_n^* t_n) \leq K \operatorname{rank}(t_n^* t_n)\,.$ Then $tr(t_n^* t_n) \leq K \operatorname{rank}(t_n^* t_n)$. Hence the lemma follows. □

Let $i_\mu : \mathbb{C}\Gamma \to \mathcal{M}_\mu$ be defined as $\rho \circ \psi$, where $\psi = \oplus_{n=1}^\infty \psi_n$ as in Subsection 6.1 and

$$\rho : B(\oplus_{n=1}^\infty \operatorname{Mat}_{V(G_n)\times V(G_n)}(\mathbb{C})) \to \mathcal{M}_\mu$$

be the quotient map from the bounded part of the direct product. The map i_μ is trace-preserving and extends to an injective trace-preserving map $\overline{i_\mu} : \mathcal{N}(\Gamma) \to \mathcal{M}_\mu$ (see [4] and [15]). The map f_3 is just i_μ composed by the embedding of $\mathcal{M}_\mu$ into its Ore-extension. We prove that f_3 is rank-preserving later.
Now let us define the map f_6. Let τ be the map defined in Subsection 8.2. Then let $j : \rho \circ \tau : \mathcal{A}_\phi \to \mathcal{M}_\mu$ and let $s = u \circ j$, where $u : \mathcal{M}_\mu \to U(\mathcal{M}_\mu)$ be the natural embedding. Then f_6 is defined as the extension of s onto $\overline{\mathcal{A}_\phi}$. We need to show of course that j is rank-preserving that is

$$\operatorname{rk}_1[\tau(\underline{a})]_{\mathcal{M}_\mu} = \operatorname{rk}_2[\tau(\underline{a})]_{\mathcal{M}_\mu^{alg}}\,,$$

for any $\underline{a} \in \oplus_{A\in Z_k} \operatorname{Mat}_{A\times A}(\mathbb{C})\,.$ However, this immediately follows from Lemma 8.2.
Now we prove the second indentity. This also shows that f_3 is rank-preserving. Again, it is enough to show that

$$[\oplus_{n=1}^\infty \psi_n(a) - \psi'_n(a))]_{\mathcal{M}_\mu} = 0\,. \tag{6}$$

We already proved that

$$\lim_{n\to\infty} \frac{\operatorname{rank}(\psi_n(a) - \psi'_n(a))}{|V(G_n)|} = 0\,.$$

Obviously, $\sup \|\psi_n(a) - \psi'_n(a)\| < \infty\,,$ hence (6) follows from Lemma 8.3.

8.4 The third identity

By definition, $i_\mu = \overline{i_\mu} \circ i$, where i is the natural embedding of $\mathbb{C}\Gamma$ into $\mathcal{N}(\Gamma)$. This immediately proves the third identity. This completes the proof of our main theorem. □

References

[1] S. K. Berberian, Baer $*$-rings. Die Grundlehren der mathematischen Wissenschaften, Band 195. Springer-Verlag, New York-Berlin, (1972)

[2] S. K. Berberian, The maximal ring of quotients of a finite von Neumann algebra. *Rocky Mount. J. Math.* **12** (1982) no. 1, 149–164.

[3] G. Elek, The rank of finitely generated modules over group algebras. *Proc. Amer. Math. Soc.* **131** (2003) no. 11, 3477–3485.

[4] G. Elek and E. Szabó, Hyperlinearity, essentially free actions and L^2-invariants. The sofic property. *Math. Ann.* **332** (2005), no. 2, 421–441.

[5] G. Elek, On the analytic zero divisor conjecture of Linnell. *Bull. Lond. Math. Soc* **35** (2003) no 2., 236–238.

[6] G. Elek and E. Szabó, Sofic groups and direct finiteness. *Journal of Algebra* 280 (2004) no.2, 426–434.

[7] G. Elek, The strong approximation conjecture holds for amenable groups. *Journal of Funct. Anal.* **239** (2006) no. 1, 345–355.

[8] K. R. Goodearl, von Neumann regular rings. *Robert E. Krieger Publishing Co.*, Inc., Malabar, FL, (1991)

[9] I. Halperin, Extensions of the rank function. *Studia Math.* **27** (1966) 325–335.

[10] I. Kaplansky, Rings of operators. *W. A. Benjamin*, Inc., New York-Amsterdam (1968)

[11] P. Linnell, Embedding group algebras into finite von Neumann regular rings. *Modules and Comodules, Trends in Mathematics, Birkhauser Verlag*, 295–300.

[12] P. Linnell and T. Schick, The Atiyah conjecture and Artinian rings *electronic* http://arxiv.org/pdf/0711.3328

[13] W. Lück, Approximating L^2-invariants by their finite-dimensional analogues. *Geom. Funct. Anal.* **4** (1994) no. 4, 455–481.

[14] D. S. Ornstein and B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, *J. Anal. Math* **48** (1987) 1-141.

[15] V. G. Pestov, Hyperlinear and sofic groups: a brief guide. *Bull. Symb. Logic* **14** (2008), no. 4, 449–480.

[16] H. Reich, On the K- and L-theory of the algebra of operators affiliated to a finite von Neumann algebra. *K-theory* **24** (2001) no.4 303–326.

[17] A. Thom, L^2-cohomology for von Neumann algebras. *GAFA* **18** (2008) no. 1 251–270.